Actuary Probability I

Transformation and Simulation of Random Variables

Irving Gómez Méndez

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UNIVERSIDAD IBEROAMERICANA
CIUDAD DE MÉXICO ®

Transformation of Random Variables

Introduction

We know the probability distribution of a random variable and are interested in determining the distribution of some function of it. For instance, suppose that we know the distribution of X and want to find the distribution of $g(X)$. To do so, it is necessary to express the event that $g(X) \leq y$ in terms of X being in some set.

Example

Let $X \sim \mathcal{U}(0,1)$. We obtain the distribution of the random variable Y, defined by $Y = X^n$, as follows: For $0 < y < 1$,

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) \\ &= \mathbb{P}(X^n \leq y) \\ &= \mathbb{P}(X \leq y^{1/n}) \\ &= F_X(y^{1/n}) \\ &= y^{1/n} \end{aligned}$$

For instance, the density function of Y is given by

$$f_Y(y) = \begin{cases} \frac{1}{n} y^{1/n-1} & \text{if } 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

That is $Y \sim \text{Beta}(1/n, 1)$.

Let X be a continuous random variable with distribution function F_X. Suppose that $g(x)$ is an increasing function . Then the distribution function of the random variable $Y = g(X)$ is given by

$$F_Y(y) = F_X(g^{-1}(x)).$$

Proof

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y) \\ &= \mathbb{P}(X \leq g^{-1}(y)) = F_X(g^{-1}(y)). \end{aligned}$$

If g is a decreasing function. Then the distribution function of the random variable $Y = g(X)$ is given by

$$F_Y(y) = 1 - F_X(g^{-1}(x)).$$

Proof

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y) \\ &= \mathbb{P}(X \geq g^{-1}(y)) = 1 - F_X(g^{-1}(y)). \end{aligned}$$

Moreover, if X has density f_X, then a simple differentiation shows that the density of the random variable $Y = g(X)$ is given by

$$f_Y(y) = \left| \frac{d}{dy} g^{-1}(y) \right| f_X(g^{-1}(y)).$$

Examples

1. Let $X \sim \mathcal{U}(0,1)$ and consider the random variable $Y = g(X) = -\frac{1}{\lambda}\log(X)$, with $\lambda > 0$. In this case $g^{-1}(y) = e^{-}\lambda y$ and the density function of Y is given by

$$\begin{aligned} f_Y(y) &= \lambda e^{-\lambda y} f_X(e^{-\lambda y}) \\ &= \lambda e^{-\lambda y} \mathbb{1}_{e^{-\lambda y} \in (0,1)} \\ &= \lambda e^{-\lambda y} \mathbb{1}_{-\lambda y \in (-\infty,0)} \\ &= \lambda e^{-\lambda y} \mathbb{1}_{y \in (0,\infty)}. \end{aligned}$$

That is, $Y \sim \text{Exp}(\lambda)$.

2. Let $T \sim \text{Weibull}(\eta, \beta)$, with density

$$f(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} e^{-(t/\eta)^\beta}\,\mathbb{1}_{t\geq 0}$$

and distribution function

$$F(t) = 1 - e^{-(t/\eta)^\beta},$$

where $\eta, \beta > 0$. Consider the random variable $Y = g(T) = \left(\frac{T}{\eta}\right)^\beta$. In this case $g^{-1}(y) = \eta y^{1/\beta}$, note that g is an increasing function. Hence, the distribution function of Y is given by

$$\begin{aligned} F(y) &= 1 - e^{-(g^{-1}(y)/\eta)^\beta} \\ &= 1 - e^{-y} \end{aligned}$$

which is the distribution function of a random variable with distribution $\text{Exp}(1)$.

We arrive to the same conclusion calculating the density of Y, for this, note that

$$\frac{d}{dy}g^{-1}(y) = \frac{\eta}{\beta}y^{\frac{1}{\beta}-1}$$

so the density of Y is given by

$$\begin{aligned}
f(y) &= \left(\frac{d}{dy}g^{-1}(y)\right)\frac{\beta}{\eta}\left(\frac{g^{-1}(y)}{\eta}\right)^{\beta-1} e^{-(g^{-1}(y)/\eta)^{\beta}}\mathbb{1}_{g^{-1}(y)\geq 0} \\
&= \frac{\eta}{\beta}y^{\frac{1}{\beta}-1}\frac{\beta}{\eta}\left(\frac{\eta y^{\frac{1}{\beta}}}{\eta}\right)^{\beta-1}\exp\left\{-\left(\frac{\eta y^{1/\beta}}{\eta}\right)^{\beta}\right\}\mathbb{1}_{\eta y^{1/\beta}\geq 0} \\
&= y^{\frac{1}{\beta}-1}y^{1-\frac{1}{\beta}}e^{-y}\mathbb{1}_{y\geq 0} \\
&= e^{-y}\mathbb{1}_{y\geq 0}.
\end{aligned}$$

That is, $Y \sim \mathrm{Exp}(1)$.

Simulation of Random Variables

Inverse Transformation Method

Let F be an increasing distribution function in x such that $0 < F(x) < 1$ and let $U \sim \mathcal{U}(0,1)$. Then the variable $Z = F^{-1}(U)$ has density F.

Proof

In this case $Z = g(U) = F^{-1}(U)$, so $g^{-1}(z) = F(z)$ and the distribution function of Z is given by

$$F_Z(z) = F_U(F(z)) = F(z),$$

hence $Z \sim F$. ■

The previous result is true in general if we use the generalized inverse $F^{\leftarrow}$ of the function F when there is not inverse, defined as

$$F^{\leftarrow}(y) = \inf\{x,\ \text{s.t. } F(x) \geq y\}.$$

That is $F^{\leftarrow}$ is the quantile function of the distribution.

These results give us a method to simulate a random variable with distribution function F. We generate a value u form a uniform variable in $(0, 1)$ and evaluate the generalized inverse in u: $F^{\leftarrow}(u)$.